UDC 62-50

ON THE NECESSITY OF A SUFFICIENT OPTIMALITY CONDITION FOR PURSUIT TIME

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P. B. GUSIATNIKOV
(Moscow)
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A necessary and sufficient condition for the optimality of the upper layer time is derived for one class of linear pursuit problems satisfying local convexity conditions.

1. Let a linear pursuit problem in an n-dimensional Euclidean space R be described by the linear vector differential equation [1 — 5]

$$dz / dt = Cz - u + v \tag{1.1}$$

(C is a constant nth - order square matrix, $u = u(t) \in P$ and $v = v(t) \in Q$ are vector-valued functions, measurable for $t \geqslant 0$, called the players' controls, $P \subset R$ and $Q \subset R$ are convex compacta) and by the terminal set $M = M_0 + W_0$, where M_0 is a linear subspace of space R and W_0 is a compact convex set in a subspace L which is the orthogonal complement to M_0 in R. By π we denote the operator of orthogonal projection onto L (we assume that $\nu = \dim L \geqslant 2$), by K the unit sphere in L, by $\Phi(t)$ the matrix e^{tC} and by $(a \cdot b)$ the scalar product of vectors $a \in R$ and $b \in R$. Let T_0 be some fixed positive number. We assume that Conditions 1 — 3 in [3] (whose notation, together with that in [4], we retain in the present paper) are fulfilled for problem (1.1); we require the fulfilment of Condition 1 only with respect to $r \in (0, T_0] = I_0$ and of Condition 3 only with respect to $t \in [0, T_0]$. By M_1 and M_2 we denote linear subspaces in R and by p_0 and q_0, vectors from R such that the linear manifolds $M_1 + p_0$ and $M_2 + q_0$ are carrier manifolds for P and Q, respectively. We set $P_0 = P - p_0$ and $Q_1 = Q - q_0$.

Condition 4. There exist a linear homeomorphism $A : M_2 \to M_1$ depending analytically on $r \in I_0$, a linear homeomorphism $\Pi(r)$: $M_1 \to L$ and the functions $f(r)$ and $g(r)$, analytic in $r \in (-\infty, +\infty)$ and positive on I_0, such that

$$\pi(r)u \equiv f(r)\Pi(r)u^* + p_0(r), \quad \pi(r)v \equiv g(r)\Pi(r)Av^* + q_0(r) \tag{1.2}$$
$$\pi(r) \equiv \pi\Phi(r), \quad u^* = u - p_0 \in P_0, \quad v^* = v - q_0 \in Q_1$$
$$p_0(r) = \pi(r)p_0, \quad q_0(r) = \pi(r)q_0 \quad \forall\, u \in P, \quad v \in Q, \quad r \in I_0$$

From relations (1.2) it follows that the boundaries of sets $P_0 \subset M_1$ and $Q_0 = AQ_1 \subset M_1$ are surfaces locally convex in M_1, and, if $\psi \in K_1$ (where K_1 is the unit sphere in M_1) and $p(\psi)$ and $q(\psi)$ are vectors maximizing the expressions $(\psi\cdot p)$, $p\in P_0$, and $(\psi\cdot q)$, $q\in Q_0$, respectively, then vectors $p(\psi)$ and $q(\psi)$ are unique and

$$u(r,\varphi)\equiv p(\Gamma(r,\varphi))+p_0,\quad v(r,\varphi)\equiv A^{-1}q(\Gamma(r,\varphi))+q_0 \qquad (1.3)$$
$$\Gamma(r,\varphi)\equiv\Pi^*(r)\varphi/|\Pi^*(r)\varphi|,\quad \Pi^*(r):L\to M_1$$
$$\forall\varphi\in K,\quad r\in I_0$$

Here $\Pi^*(r)$ is a linear homeomorphism depending analytically on $r\in I_0$ adjoint to $\Pi(r)$, i.e., giving the equality

$$(x\cdot\Pi(r)y)\equiv(\Pi^*(r)x\cdot y),\quad \forall r\in I_0,\quad x\in L,\quad y\in M_1$$

Let

$$w_*(r)=\pi(r)P \overset{*}{-} \pi(r)Q,\quad \overline{w}(r)=f(r)P_0 \overset{*}{-} g(r)Q_0$$

Then (see [7, 8])

$$w_*(r)\equiv\Pi(r)\overline{w}(r)+\Delta(r),\quad \Delta(r)=p_0(r)-q_0(r)$$

It is well known [9] that when Conditions 1 – 4 are fulfilled the condition of total sweep

$$\overline{w}(r)+g(r)Q_0\equiv f(r)P_0,\quad r\in I_0 \qquad (1.4)$$

is sufficient for the global [4] optimality of time $T(z)\leqslant T_0$, constructed in [5].

Condition 5. There exist a ν-dimensional linear subspace $M_3\subset R$, a linear homeomorphism $B:M_3\to M_1$ and a function $k(r)$ analytic in $r\in(-\infty,+\infty)$, such that the triple $\varkappa=\{f(r),\ g(r),\ k(r)\}$ is linearly independent on I_0 and such that

$$\pi(t)w=k(t)\,\Pi(t)Bw,\quad \forall t\in I_0,\quad w\in M_3$$

2. Theorem 1. Let Conditions 1 – 5 be fulfilled for problem (1.1). Then the total sweep condition is a necessary condition for the global optimality of time $T(z)\leqslant T_0$.

The proof of Theorem 1 is carried out in several stages and is based on Theorem 2 in [8].

3. We set

$$p(\varphi,\psi)=(\varphi\cdot p(\varphi)-p(\psi)),\quad q(\varphi,\psi)=(\varphi\cdot q(\varphi)-q(\psi))$$
$$h(\varphi,\psi)=q(\varphi,\psi)/p(\varphi,\psi),\quad \alpha=\sup h(\varphi,\psi)$$

(the sup is taken over all $\varphi,\psi\in K_1$, $\varphi\neq\psi$). In [8] it was shown that a point φ_0 and a local coordinate system $\bar{s}=(s^2,\ldots,s^\nu)$ in its neighborhood $O_{\varphi_0}\subset K_1$ with origin O at point φ_0 exist such that

$$\varphi = \varphi(\bar{s}) = \varphi(s^2, \ldots, s^\nu), \quad \varphi \in O_{\varphi_0}, \quad \varphi(0) = \varphi_0 \tag{3.1}$$
$$q_{22}(\varphi(0)) = \alpha p_{22}(\varphi(0))$$
$$q_{ij}(\varphi(\bar{s})) = \left(\varphi_i(\bar{s}) \cdot \frac{\partial q(\varphi(\bar{s}))}{\partial s^j}\right), \quad p_{ij}(\varphi(\bar{s})) = \left(\varphi_i(\bar{s}) \cdot \frac{\partial p(\varphi(\bar{s}))}{\partial s^j}\right);$$
$$\varphi_i(\bar{s}) = \frac{\partial \varphi(\bar{s})}{\partial s^i}, \quad i, j = 2, \ldots, \nu.$$

Also in [8] it was proved that the total sweep (1.4) obtains if and only if

$$m(r) \geqslant 1, \quad m(r) = f(r) / (\alpha g(r)), \quad r \in I_0$$

Assumption 1. There exist $0 < \tau < \tau_1 \leqslant T_0$ such that $m(r) \geqslant 1$, $r \in (0, \tau]$, and $m(r) < 1$, $r \in (\tau, \tau_1]$.

Note 1. Because $m(r)$ is analytic we can find $\tau_2 \in (\tau, \tau_1)$, such that $m'(r) < 0$, $r \in \Gamma \equiv (\tau, \tau_2]$.

It will be shown in Paragraphs 4 – 6 that when Assumption 1 and the hypotheses of Theorem 1 are fulfilled we can find a point z_*, in space R, for which the time $T(z_*) < T_0$ is not optimal.

4. Lemma 1. Let $\theta \in (\tau, \tau_2)$. Then for any sufficiently small $\tau_0 \in (0, \tau)$, $\theta + \tau_0 \in \Gamma$, the determinant $\Delta = \Delta_1(\theta + \tau_0) \neq 0$ (here $\Delta_1(t)$ is the Wronskian for the system of functions $f(t)$, $g(t)$ and $k(t)$) and the function

$$R(t) = f(t + \tau_0)\, g(\theta + \tau_0) - f(\theta + \tau_0)\, g(t + \tau_0)$$

satisfies the following relations:

$$R(t) > 0, \quad t \in [0, \theta), \quad R(\theta) = 0, \quad -R'(\theta) = N > 0 \tag{4.1}$$

By virtue of the analyticity of the functions occurring in triple $\varkappa$, the first part of the lemma follows [10] from the linear independence of these functions. The second part follows from Assumption 1, Note 1 and the representation

$$R(t) = \alpha g(\theta + \tau_0) g(t + \tau_0)(m(t + \tau_0) - m(\theta + \tau_0))$$

Corollary 1. For any sufficiently small $\tau_0 > 0$ there exist analytic functions $h_1(t)$, $h_2(t)$ and $H(t) = h_3(t)$ each being a linear combination of functions $f(t + \tau_0)$, $g(t + \tau_0)$ and $k(t + \tau_0)$, satisfying the conditions

$$d^j h_i(\theta) / dt^j = \begin{cases} 0, & j \neq i - 1 \\ 1, & j = i - 1 \end{cases}; \quad j = 0, 1, 2; \; i = 1, 2, 3 \tag{4.2}$$

To verify the corollary it is enough to note that by virtue of Lemma 1 we have a linear system with determinant $\Delta \neq 0$ for finding the coefficients of each linear combination.

Everywhere below we fix $\theta \in (\tau, \tau_2)$ and the number $\tau_0 > 0$ so small that the conclusion of Lemma 1 is satisfied. We set

$$L(t) = \Pi(t + \tau_0), \quad D(t, \varphi) = (L^{-1}(t))^* \varphi \, / \, |(L^{-1}(t))^* \varphi|$$
$$M(t, \varphi) = L^{-1}(t) W(t, D(t, \varphi)), \quad C(t) z = L^{-1}(t) \pi(t) z$$
$$\forall t \in [0, \theta] = I_1, \quad \varphi \in K_1, \quad z \in R$$

Here $L^{-1}(t) : L \to M_1$ is the operator inverse to $L(t)$, the sign $*$ denotes passage to the adjoint operator; as is well known, $(L^{-1}(t))^* = (L^*(t))^{-1}$. Operator $L(t)$ is nonsingular for each $t \in I_1$; therefore, operator $L^*(t)$ is nonsingular too and the family of surfaces $M(t, K_1)$, $t \in I_1$, is locally convex [5]. In connection with this there exists $c_2 > 0$ such that (see Lemma 2 in [5])

$$(\varphi \cdot M(t, \varphi) - M(t, \psi)) \geqslant c_2(\varphi \cdot \varphi - \psi)$$
$$\forall t \in [\tau, \theta], \quad \varphi \in K_1, \quad \psi \in K_1$$

We remark that the representation for $M(t, \varphi)$ has been chosen so that the vector φ is the outward normal to surface $M(t, K_1)$ at point $M(t, \varphi)$.

N o t e 2. Since

$$(\psi \cdot W(t, \psi) - \pi(t) z) = (L^*(t)\psi \cdot L^{-1}(t) W(t, \psi) - C(t) z) =$$
$$l(t, \varphi)(\varphi \cdot M(t, \varphi) - C(t) z),$$
$$\varphi = L^*(t)\psi \, / \, |L^*(t)\psi| \in K_1, \quad l(t, \varphi) = |(L^{-1}(t))^* \varphi|^{-1};$$
$$\forall \psi \in K, \; z \in R, \; t \in I_1$$

function $\lambda(z, t)$ has the same sign and the same zeros as the function

$$n(z, t) = \min_{\varphi \in K_1} (\varphi \cdot M(t, \varphi) - C(t) z) \tag{4.3}$$

We denote $\psi_*(z, t) \equiv L^*(t)\psi(z, t) \, / \, |L^*(t)\psi(z, t)|$ (vector $\psi(z, t)$ was introduced in [4])(*). Then, if $\varphi(z, t)$ is the vector giving the minimum in (4.3) and if $\lambda(z, t) = 0$, then $\varphi(z, t) = \psi_*(z, t)$.

N o t e 3. Let $\varphi_0 = \varphi(0)$ be the vector from (3.1). By virtue of Corollary 1 and Conditions 4 and 5, a vector $z_0 \in R$ exists such that

$$C(t) z_0 = M(\theta, \varphi_1) h_1(t) + \frac{\partial M(\theta, \varphi_1)}{\partial t} h_2(t) + \frac{\partial^2 M(\theta, \varphi_1)}{\partial t^2} h_3(t), \quad t \geqslant 0 \tag{4.4}$$

So that, with due regard to (4.2), $M(t, \varphi_1) - C(t) z_0 = \varepsilon(t)$, $|\varepsilon(t)| \leqslant c_0^* (\theta - t)^3$, $0 \leqslant t \leqslant \theta$, where $c_0^* > 0$ is some fixed constant. For any real a, b and c a vector $z^*(a, b, c) \in R$ exists yielding the equality

*) Editor's Note. In the English edition this vector is introduced in Lemma 1 on p. 193, PMM Vol. 37, No. 2, 1973.

$$C\ (t)z^*(a,\ b,\ c) = (aR\ (t) + bH\ (t))\varphi_1 + cH\ (t)\chi_1, \quad t \geqslant 0 \tag{4.5}$$

$$\varphi_1 = \omega(\theta,\ \varphi_0) \in K_1, \quad \omega(r,\ \varphi) \equiv \frac{N^{-1}(r)\,\varphi}{|N^{-1}(r)\,\varphi|}, \quad \varphi \in K_1, \quad r \in I_0$$

$$\chi_1 = \frac{\partial}{\partial s^2} M\ (\theta,\ \omega\ (\theta,\ \varphi\ (0))), \quad \psi_0 = D\ (\theta,\ \varphi_1), \quad N\ (r) \equiv \Pi^*\ (r)\ (L^*\ (r))^{-1}$$

Here χ_1 is a nonzero vector orthogonal to φ_1 (by expanding, if necessary, the local coordinates we can assume that $|\chi_1| = 1$).

Let us clarify Note 3. The right hand side of each of the equalities (4.4) and (4.5) has the form

$$f\ (t + \tau_0)u_0 + g\ (t + \tau_0)Av_0 + k\ (t + \tau_0)Bw_0,$$
$$u_0 \in M_1, \quad v_0 \in M_2, \quad w_0 \in M_3$$

Therefore, it is sufficient to take the vector $z = e^{\tau_0 C}\ (u_0 + v_0 + w_0)$ in the left hand side. Notice also that the mapping $N(r)\,\varphi$ is analytic in $r \in (0,\ \theta]$, $\varphi \in K_1$, so that we can find $c_3 > 0$ such that

$$|\ N\ (r)\varphi - N\ (\theta)\varphi\ | \leqslant c_3(\theta - r), \quad r \in [\tau,\ \theta], \quad \varphi \in K_1 \tag{4.6}$$

We set $z\ (a,\ b,\ c) = z_0 + z^*(a,\ b,\ c)$; $\theta\ (t) = \theta - t$. We have

$$\pi\ (\theta)z\ (a,\ b,\ c) = W\ (\theta,\ \psi_0), \quad \psi\ (z\ (a,\ b,\ c),\ \theta) = \psi_0 \tag{4.7}$$

5. By $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_m < \theta$ we denote all the zeros of function $H\ (t)$ in the half-open interval $[0,\ \theta)$ and by $\theta_* > \tau$, a fixed number $\theta_* \in (\theta_m,\ \theta)$ so close to θ that

$$\theta^2\,(t) < 4H\ (t) \leqslant 4\theta^2\,(t), \quad 1 \leqslant \frac{2R\ (t)}{N\theta\ (t)} \leqslant 2 \tag{5.1}$$

$$8\ |\ \varepsilon\ (t)\ | \leqslant c_2(\theta\ (t))^{5/2} \leqslant c_2\ /\ 16, \quad \forall\ t \in I = [\theta_*,\ \theta) \subset (\tau,\ \theta)$$

We set

$$E = \max_{t \in I_1,\ \varphi \in K_1} (|\ C\ (t)\ z_0\ | + |\ M\ (t,\ \varphi)\ |), \quad Y = \min_{t \in [0,\ \theta_*]} R\ (t) > 0 \tag{5.2}$$

$$a_0 = 2Y^{-1}(E + 2^8\ E^2N^2(c_2Y^2)^{-1} + 4c_2), \quad \theta_0 = \theta - \delta_0$$

$$\delta_0 = \min\{\theta - \theta_*,\ Y^3 2^{-7} N^{-3},\ (4a_0 N c_2^{-1})^{2/3},\ c_2{}^2 Y^3 4^{-7} E^{-2} N^{-3}\}$$

$$a_1 = 2a_0(N + Y) + (32EN)^2(c_2Y^2)^{-1} + 4c_2$$

L e m m a 2. For any $T \in I^0 = (\theta_0,\ \theta)$ we can find numbers $a = a\ (T) \equiv a_0$, $b = b\ (T)$, $c = c\ (T) \equiv 4E\ (\theta - T)^{-2}$ and a nonempty set $\Omega\ (T)$ whose closure is contained in interval $(T,\ \theta)$, such that:

a) $\lambda\ (z\ (a,\ b,\ c),\ t) < 0$, $t \in [0,\ T] = X$; $\lambda\ (z\ (a,\ b,\ c),\ t) \leqslant 0$, $t \in [T,\ \theta]$;

b) if $\lambda\ (z\ (a,\ b,\ c),\ t) = 0$ and $t \in [0,\ \theta)$, then $t \in \Omega\ (T)$, and vice versa;

c) $| aR(t) + bH(t) | \leqslant a_1(\theta - T)^{3/2}$; $| cH(t) | \leqslant 4E,\ t \in [T, \theta]$.

Proof. We set

$$b^* = b^*(T) = -\frac{aR(r)}{H(r)} - \frac{64E^2H(r)}{c_2\theta^4(T)} - 4c_2(\theta(r))^{1/2},\quad r = \theta - \frac{4N(\theta(T))^{4/3}}{Y} > T \quad (5.3)$$

$$T^* = \max\{(\theta + r)/2,\ \theta - a_0N(2|b^*| + c_2)^{-1}\} \quad (5.4)$$

For any $\bar{b} \in [b^*, 0]$ we denote the vector $z(a, \bar{b}, c)$, by $z(\bar{b})$, where $a \equiv a(T)$ and $c \equiv c(T)$ are specified by Lemma 2. Then

$$\lambda(z(\bar{b}), t) < 0,\quad t \in X,\ \bar{b} \in [b^*, 0] \quad (5.5)$$

Indeed, using the orthogonality of φ_1 and χ_1 and relations (4.5) and (5.2), we have $(\sigma(t) = \operatorname{sign} H(t))$

$$n(z(\bar{b}), t) \leqslant (\sigma(t)\chi_1\cdot M(t, \sigma(t)\chi_1) - C(t)z(\bar{b})) = (\sigma(t)\chi_1\cdot M(t, \sigma(t)\chi_1) - C(t)z_0) - c|H(t)| \leqslant E - 4E|H(t)|(\theta - T)^{-2} < 0$$

for those $t \in X$ for which $4|H(t)| > (\theta - T)^2$. By virtue of (5.1) we have the inclusion $t \in [0, \theta_*]$ for those $t \in X$ for which $4|H(t)| \leqslant (\theta - T)^2$. So that, using (5.1) – (5.3) and the inequality $\theta - T < 1$, we obtain, as in [8],

$$n(z(\bar{b}), t) \leqslant (\varphi_1\cdot M(t, \varphi_1) - C(t)z_0) - (aR(t) + \bar{b}H(t)) \leqslant E - a_0Y + |b^*|\theta^2(T)/4 < 0$$

Inequality (5.5) has been proved (see Note 2).

Let us show that

$$\lambda(z(\bar{b}), t) < 0,\ t \in [T^*, \theta],\ t \neq \theta,\ \bar{b} \in [b^*, 0] \quad (5.6)$$

Indeed, $n(z(\bar{b}), t) \leqslant |\varepsilon(t)| - a_0R(t) + |b^*|H(t) < 0,\ t \in [T^*, \theta)$. Let us prove the inequality

$$\lambda(z(b^*), r) > 0 \quad (5.7)$$

We set $n_* = n(z(b^*), r)$; $l_* = aR(r) + b^*H(r)$. By virtue of (5.1) – (5.3)

$$c_2 > c_2 + l_* = c_2 - 64E^2H^2(r)\,\theta^{-4}(T)\,c_2^{-1} - 4c_2\theta^{1/2}(r)\,H(r) > {}^1/_2c_2 \quad (5.8)$$

Therefore, for the quantity $n_* = (\varphi\cdot M(r, \varphi) - M(r, \varphi_1) + \varepsilon(r) - l_*\varphi_1 - cH(r)\chi_1)$, where $\varphi = \varphi(z(b^*), r)$, we have the estimate

$$n_* \geqslant c_2(\varphi\cdot\varphi - \varphi_1) - 8^{-1}c_2(\theta - r)^{5/2} - (\varphi\cdot l_*\varphi_1 + cH(r)\chi_1) \geqslant c_2 - 8^{-1}c_2(\theta(r))^{5/2} - [(c_2 + l_*)^2 + c^2H^2(r)]^{1/2}$$

Hence from (5.8) we obtain

$$n_* \geqslant -l_* - 8^{-1}c_2(\theta(r))^{5/2} - c^2H^2(r)\,c_2^{-1} > 0$$

By virtue of Note 2, inequality (5.7) is proved.

Finally, let us show that

$$\lambda(z(0), t) \equiv \lambda(z(a, 0, c), t) < 0,\ t \in [0, \theta)$$

In accord with (5.5) it suffices to verify this only for $t \in [T, \theta)$. For such t we have

$$n(z(0),t) \leqslant (\varphi_1 \cdot M(t, \varphi_1) - C(t) z_0 - a_0 R(t) \varphi_1) \leqslant |\varepsilon(t)| - a_0 R(t) < 0$$

as required.

Let us complete the proof of Lemma 2. Let $b(T)$ be the least upper bound of the set of all $\bar{b} \in [b^*, 0]$ for which the function $\lambda(z(\bar{b}), t)$ vanishes at least at one point of the interval $t \in (T, T^*)$. Then relations a) and b) of Lemma 2 are fulfilled, while estimate c) follows from (5.1) – (5.3)

$$|a(T) R(t) + b(T) H(t)| \leqslant a_0 R(t) + |b^* H(t)| \leqslant a_1 (\theta - T)^{2/3}$$
$$t \in [T, \theta)$$

6. Using Assumption 1 we complete the proof of Theorem 1. Let $T_i \to \theta - 0$, $i \to \infty$. By z_i we denote the point $z(a(T_i), b(T_i), c(T_i))$ (see Lemma 2), by $l_i(t)$ and $c_i(t)$ the functions $a(T_i)R(t) + b(T_i)H(t)$ and $c(T_i)H(t)$, by Ω_i the set $\Omega(T_i)$. If $t \in \Omega_i$, we denote the vector $\Gamma(t, \psi(z_i, t))$ by φ_{it}. By virtue of Note 2, when $t \in \Omega_i$ we have

$$M_i(t) \equiv M(t, \omega(t, \varphi_{it})) = C(t) z_i = M(t, \omega(\theta, \varphi_0)) + l_i(t)\varphi_1 + c_i(t)\chi_1 - \varepsilon(t) \qquad (6.1)$$

Multiplying (6.1) scalarly by φ_1 and using the local convexity of $M(t, \varphi)$, we obtain,

$$0 \leqslant c_2(\varphi_1 \cdot \varphi_1 - \omega(t, \varphi_{it})) \leqslant (\varphi_1 \cdot M(t, \varphi_1) - M_i(t)) = -l_i(t) + (\varphi_1 \cdot \varepsilon(t)) = c_2 k_i^2(t) \qquad (6.2)$$

Having made use of the inequality $|a|a|^{-1} - b|b|^{-1}|^2 \leqslant |a - b|^2 \cdot (|a|\,|b|)^{-1}$, when $t \in \Omega_i$ we have (by virtue of (6.2), (4.6), (5.1) and estimate c) in Lemma 2)

$$|\varphi_0 - \varphi_{it}|^2 \leqslant (|N(\theta)\varphi_1|\,|N(t)\,\omega(t, \varphi_{it})|)^{-1} |N(\theta)\varphi_1 - N(t)\varphi_1 + N(t)(\varphi_1 - \omega(t, \varphi_{it}))|^2 \leqslant N_1^2[c_3\theta(t) + 2N_0 k_i(t)]^2 \leqslant \{N_1(c_3 + N_0(2a_1 / c_2 + 1))\}^2(\theta - T_i)^{2/3} \qquad (6.3)$$
$$N_1 = \sup \|N^{-1}(t)\|, \ t \in I^0; \quad N_0 = \sup \|N(t)\|, \ t \in I^0$$

Here $\|\cdot\|$ denotes the norms of the corresponding linear operators. Therefore, for all sufficiently large i (for all $i = 1, 2, \ldots$) if we discard a finite number of terms)

$$\varphi_{it} = \varphi(\bar{\sigma}_{it}) \in Q_{\varphi_0}, \ N(\theta)\,\omega(t, \varphi_{it}) / |N(\theta)\,\omega(t, \varphi_{it})| = \varphi(\bar{s}_{it}) \in O_{\varphi_0}$$

where $\bar{\sigma}_{it}$ and $\bar{s}_{it}$ are the local coordinates of the corresponding vectors. By virtue of (6.3) a sequence $\varepsilon_i \to +0$, $i \to \infty$, exists such that for any i and all $t \in \Omega_i$

$$|\bar{s}_{it}| \leqslant \varepsilon_i, \quad |\bar{\sigma}_{it}| \leqslant \varepsilon_i \tag{6.4}$$

From the Taylor expansion with a remainder term in Lagrange form follows

$$M(r, \omega(\theta, \varphi(\bar{s}))) = M(r, \varphi_I) + \sum_{j=2}^{\nu} \frac{\partial M(r, \omega(\theta, \varphi(0)))}{\partial s^j} s^j + O(|\bar{s}|^2) \tag{6.5}$$

$$|O(|\bar{s}|^2)| \leqslant c_0 |\bar{s}|^2$$

Here c_0 is the common constant for all $r \in [\theta_*, \theta]$ and all $\varphi(\bar{s}) \in O_{\varphi_0}$. From (6.1) and (6.5) follows

$$l_i(t)\varphi_I + c_i(t)\chi_I - \varepsilon(t) = \sum_{j=2}^{\nu} \frac{\partial M(t, \omega(\theta, \varphi(0)))}{\partial s^j} s_{it}^j + O(|\bar{s}_{it}|^2) \tag{6.6}$$

$$t \in \Omega_i$$

Multiplying (6.6) scalarly by $\partial\omega(\theta, \varphi(0)) / \partial s^k$, we obtain

$$c_i(t) M_{k2}(\theta) + \varepsilon_k(t) = \sum_{j=2}^{\nu} M_{kj}(t) s_{it}^j + \Delta_{ki}(t); \; t \in \Omega_i, \quad k = 2, \ldots, \nu \tag{6.7}$$

$$|\varepsilon_k(t)| \leqslant 8^{-1} c_* (\theta - t)^{3/2}, \quad |\Delta_{ki}(t)| \leqslant c_* |\bar{s}_{it}|^2$$

$$c_* = (1 + c_0 + c_2)\left(1 + \sum_{k=2}^{\nu} \left| \frac{\partial\omega(\theta, \varphi(0))}{\partial s^k} \right|\right)$$

$$M_{kj}(t) \equiv M_{kj}(t, \varphi_0), \quad M_{kj}(t, \varphi(\bar{s})) = \left(\frac{\partial\omega(\theta, \varphi(\bar{s}))}{\partial s^k} \frac{\partial M(t, \omega(\theta, \varphi(\bar{s})))}{\partial s^j}\right) \tag{6.8}$$

$$k, j = 2, \ldots, \nu$$

Solving the equation system (6.7) relative to s_{it}^m, $m = 2, \ldots, \nu$ (the quadratic form with matrix (6.8) is positive definite, so that the matrix $B_{mk}(t)$ inverse to matrix $M_{kj}(t)$ exists and is continuous in $t \in [\theta_*, \theta]$), we have (see (5.1); δ_2^m is the Kronecker symbol)

$$s_{it}^m + \gamma_i^m(t) = c_i(t)(\delta_2^m + \xi_*^m(t)) + \varepsilon_m^*(t), \quad t \in \Omega_i \tag{6.9}$$

$$|\xi_*^m(t)| = \left|\sum_{k=2}^{\nu} B_{mk}(t) m(k, t)\right| \leqslant \bar{c} \sup_{t \in [T_i, \theta]} \sum_{k=2}^{\nu} |m(k, t)| = \delta_i \to 0$$

$$i \to \infty$$

$$|\varepsilon_m^*(t)| = \left|\sum_{k=2}^{\nu} B_{mk}(t) \varepsilon_k(t)\right| \leqslant \bar{c} H(t) (\theta - t)^{1/2} \leqslant c_i(t) \delta_i^*$$

$$|\gamma_i^m(t)| = \left|\sum_{k=2}^{\nu} B_{mk}(t) \Delta_{ki}(t)\right| \leqslant \bar{c} c_* |\bar{s}_{it}|^2$$

$$m(k,t) = M_{k2}(\theta) - M_{k2}(t),\ \bar{c} = (c_* + 1)\Big(1 + \sup_{t\in[\theta_*,\theta]} \sum_{m,k=2}^{\nu} |B_{mk}(t)|\Big)$$
$$\delta_i^* = \bar{c}\,(4E)^{-1}(\theta - T_i)^{5/2} \to 0, \quad i\to\infty$$

From relations (6.9) it follows (cf. [8]) that for all sufficiently large i

$$s_{it}^m = c_i(t)(\delta_2^m + \alpha_i^m(t)), \quad t\in\Omega_i, \quad m = 2,\ldots, \tag{6.10}$$
$$|\alpha_i^m(t)| \leqslant \delta_i + \delta_i^* + 27\nu^2\bar{c}c_*\varepsilon_i \to 0, \quad i\to\infty$$

For the determination of the local coordinates σ_{it}^m we have the relation

$$\varphi_{it} = N(t)\omega(\theta, \varphi(\bar{s}_{it}))/|N(t)\omega(\theta,\varphi(\bar{s}_{it}))| = \tag{6.11}$$
$$\varphi(\bar{s}_{it}) + \omega_i(t), \ i\in\Omega_i$$

where, as in (6.3),

$$|\omega_i(t)| \leqslant N_1c_3(\theta - t) \leqslant c_i(t)(N_1c_3E^{-1})(\theta - T_i)^2(\theta - t)^{-1}$$

By virtue of (5.4) and (5.6) we have

$$\theta - t \geqslant \theta - T_i^* = \min\{\theta - r_i,\ a_0N(2|b_i^*| + c_2)^{-1}\}$$

Taking into account the inequality

$$|b_i^*| \leqslant (\theta - r_i)^{-1}(8a_0N + 4^6E^2N^3 + c_2)$$

following from (5.3), expanding (6.11) by Taylor's formula and arguing analogously to (6.6) –(6.10), we obtain

$$\sigma_{it}^m = c_i(t)(\delta_2^m + \beta_i^m(t)), \quad t\in\Omega_i, \quad m = 2,\ldots,\nu \tag{6.12}$$
$$|\beta_i^m(t)| \leqslant \beta_i \to 0, \quad i\to\infty$$

where all the β_i depend neither on m, nor on $t\in\Omega_i$.

Let us compute (cf. Sect. 2 in [8]) the quantity $\mu_i(t) = \mu(t, \psi(z_i, t), \theta, \psi(z_i, \theta))$. By virtue of (1.3), (6.1), (4.7) and Note 2, for any $t\in\Omega_i$ we have

$$\eta_i(t) = \mu_i(t)\,|\Pi_i^*(t)\psi(z_i, t)| = f(t)(\varphi_{it}\cdot p(\varphi_{it}) - p(\varphi_0)) -$$
$$g(t)(\varphi_{it}\cdot q(\varphi_{it}) - q(\varphi_0))$$

Expanding the expression within the parentheses by Taylor's formula, we obtain

$$\eta_i(t) = \frac{1}{2} f(t) \sum_{m,k=2}^{\nu} p_{mk}(\varphi_0)\,\sigma_{it}^m\sigma_{it}^k -$$
$$\frac{1}{2} g(t) \sum_{m,k=2}^{\nu} q_{mk}(\varphi_0)\,\sigma_{it}^m\sigma_{it}^k + o_t(|\bar{\sigma}_{it}|^2), \quad t\in\Omega_i$$

where $o_t(|\bar{s}|^2)/|\bar{s}|^2 \to 0,\ |\bar{s}|\to 0$, uniformly in $t\in[\theta_*, \theta]$. Substituting the values for the local coordinates from (6.12), we have (see (3.1), inclusion $t\in[\theta_*, \theta]$ and Note 1)

$$c_i^{-2}(t)\eta_i(t) = {}^1/_2\alpha g(t)[m(t) - 1]p_{22}(\varphi(0)) + \sigma(i, t) \leqslant \qquad (6.13)$$
$${}^1/_2\alpha g^*[m(\theta_*) - 1]p_{22}(\varphi(0)) + \sigma(i, t), \quad t \in \Omega_i$$

where (see (6.12)) $g^* = \min t \in [\theta_*, \theta]\, g(t) > 0$ and $|\sigma(i, t)| \to 0,\ i \to \infty$, uniformly in $t \in \Omega_i$. Therefore, $\mu_i(t) < 0$ for any $t \in \Omega_i$ for all sufficiently large i. By virtue of assertions a) and b) in Lemma 2 this signifies that all the hypotheses of Theorem 2 in [8] have been fulfilled for point z_i. Theorem 1 is proved in Assumption 1 is fulfilled.

7. Assumption 2. There exists $0 < \tau_1 < T_0$ such that $m(r) < 1$ for $0 < r < \tau_1$.

To carry out the proof of Theorem 1 under the conditions of Assumption 2 it is sufficient to set $\tau = 0$, to choose $\tau_2 \in (\tau, \tau_1)$ such that $m'(r) = (f(r)\ /\ (\alpha g(r)))' \neq 0$ for $r \in \Gamma = (\tau, \tau_2]$ (this is possible because the functions $f(r)$ and $g(r)$ expand into power series in parameter r in a neighborhood of $r = 0$, to choose $\theta \in \Gamma$ and $\tau_0 > 0$ so as to satisfy the conclusion of Corollary 1 and the relations (4.2) and also such that the function

$$R(t) = (f(t + \tau_0)g(\theta + \tau_0) - f(\theta + \tau_0)g(t + \tau_0))\omega$$
$$\omega = \operatorname{sign} m'(s), \quad s \in \Gamma$$

satisfies (4.1), and to repeat verbatim the arguments in Sections 4 — 6 up to formula (6.13).

8. We now present an example showing that condition A in [2] in the general case is not a necessary condition for the global optimality of the upper layer time

$$dz_1\ /\ dt = z_2 - u, \quad dz_2\ /\ dt = v; \quad |u| \leqslant 1, \quad |v| \leqslant 1 \qquad (8.1)$$

where z_1, z_2, u and v are two-dimensional vectors. The terminal set M is the subspace $\{z : z_1 = 0\}$. Here $\pi z = z_1$; $W(t, \varphi) = h(t)\varphi$, $h(t) = t - t^2/2$, $0 \leqslant t \leqslant 2$. The time $T(z)$ is the smallest positive root of the equation $F(t, z) = -|z_1 + tz_2|^2 + (t - t^2/2)^2 = 0$. If $T(z) \leqslant 1$, the optimality of $T(z)$ follows from [2]. Let us show that time $T(z) \in (1, 2)$ also is optimal although condition A may not hold on the whole interval $[0,2)$.

We suggest that for escape starting from point z_0, $T(z_0) = T_0 \in (1,2)$, we set

$$\bar v(s) = (T_0 - s)^{-1}u(s) + (1 - (T_0 - s)^{-1})\varphi_0, \quad 0 \leqslant s \leqslant T_0 - 1$$
$$\bar v(s) = u(s), \quad 0 \leqslant T_0 - s \leqslant 1$$

where $\varphi_0 = \varphi(z_0)$ is given by the equality (cf. [4]) $h(T_0)\varphi_0 = z_{10} + T_0 z_{20}$. Then for the motion $z(s)$, $0 \leqslant s \leqslant T_0$, $z(0) = z_0$, we have

$$F(T_0 - s, z(s)) = -\left| z_{10} + sz_{20} + \int_0^s (s - r)\bar v(r)\,dr - \int_0^s u(r)\,dr + (T_0 - s)z_{20} + (T_0 - s)\int_0^s \bar v(r)\,dr \right|^2 + h^2(T_0 - s) = h^2(T_0 - s) - \left|\left(T_0 - s - \frac{(T_0 - s)^2}{2}\right)\varphi_0\right|^2 = 0$$

for all $s \in [0, T_0 - 1]$. Let us show that $T(z(s)) \equiv T_0 - s$ is fulfilled for all such s. We proceed by contradiction. Let $0 < T(z(s)) < T_0 - s$. By virtue of the definition of T_0 we have $0 \leqslant k = \partial F(T_0, z_0) / \partial t$, and, if $k = 0$, then $n = \partial^2 F(T_0, z_0) / \partial t^2 \leqslant 0$. Using the inequality $k \geqslant 0$, by direct calculations we obtain

$$\frac{\partial F(T_0 - s, z(s))}{\partial t} \geqslant (T_0 - s)(2 - T_0 + s)\left(s - \left(\varphi_0 \cdot \int_0^s \bar{v}(r)\, dr\right)\right) \geqslant 0$$

where equality to zero is possible only if $k = 0$ and $u(r) \equiv \varphi_0$ almost everywhere on $[0, s]$. But in the latter case

$$\frac{\partial^2 F(T_0 - s, z(s))}{\partial t^2} = n + s(2 + s - 2T_0) < 0$$

From what has been said it follows that the function $p(t) = F(t, z(s))$ has at least three zeros (with regard to their multiplicities) on the interval $(0, T_0 - s]$ if $k \neq 0$ and four zeros if $k = 0$. In the latter case we obtain a contradiction that, since $p(0) < 0$ and $p(-\infty) > 0$, a fourth-degree polynomial has five roots. However, if $k \neq 0$, then $p(t) > 0$ for all $t > T_0 - s$ sufficiently close to $T_0 - s$; $p(2) \leqslant 0$, which yields four roots on $(T_0 - s, 2]$. As before, we discover five roots on the negative semiaxis. A contradiction.

Now let $T(z(s_0)) = 0$, $s_0 \in (0, T_0 - 1]$. Without loss of generality we can take it that s_0 is the smallest one of such instants. By what has been proved, $T(z(s)) \equiv T_0 - s$, $0 \leqslant s < s_0$, so that $F(0, z(s_0)) = 0$; $F(T_0 - s_0, z(s_0)) = 0$; $F(t, z(s_0)) \leqslant 0$, $0 \leqslant t \leqslant T_0 - s_0$. Since $z_1(s_0) = 0$, $F(t, z(s_0)) = t^2(-|z_2(s_0)|^2 + (1 - t/2)^2)$.

Hence

$$|z_2(s_0)|^2 = (1 - (T_0 - s_0)/2)^2$$
$$F(t, z(s_0)) = {}^1/_4 t^2 (T_0 - s_0 - t)(2 - t + 2 - (T_0 - s_0)) > 0$$
$$0 < t < T_0 - s_0$$

A contradiction. The inequality $T(z(s)) \geqslant T_0 - s$, $0 \leqslant T_0 - s \leqslant 1$, follows from [2].

Now let $\delta > 0$. Having chosen $\varepsilon > 0$ sufficiently small and setting $v(s) \equiv \varphi_0$, $0 \leqslant s \leqslant \varepsilon$; $v(s) \equiv \bar{v}(s - \varepsilon)$, $s > \varepsilon$, we guarantee avoidance of contact during time $T_0 - \delta$ (see [2] for the proof).

9. A large class of pursuit problems satisfying Conditions 1 –5 of the present paper (remember that Conditions 1 –3 are taken from [3]) have been presented in Sect. 5 of [9]. Thus, for this class we have obtained a necessary and sufficient condition for the global optimality of first absorption time.

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REFERENCES

1. Krasovskii, N. N. and Subbotin, A. I., Optimal strategies in a linear differential game. PMM. Vol. 33, No. 4, 1969.

2. Gusiatnikov, P. B. and Nikol'skii, M. S., On the optimality of pursuit time. Dokl. Akad. Nauk SSSR, Vol. 184, No. 3, 1969.

3. G u s i a t n i k o v, P. B., Necessary optimality conditions in a linear pursuit problem. PMM Vol. 35, No. 5, 1971.

4. G u s i a t n i k o v, P. B., Necessary optimality condition for the time of first absorption. PMM Vol. 37, No. 2, 1973.

5. G u s i a t n i k o v, P. B., On a statement of linear pursuit problems. Differents. Uravn., Vol. 8, No. 8, 1972.

6. L e e, E. B. and M a r k u s, L., Fundamentals of Optimal Control Theory. Moscow, "Nauka", 1972.

7. H a d w i g e r, H., Vorlesunger über Inhalt, Oberfläche und Isoperimetrie. Berlin — Göttingen —Heidelberg, Springer — Verlag, 1957.

8. G u s i a t n i k o v, P. B., The necessity of a sufficient optimality condition for pursuit time. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, No. 1, 1978.

9. G u s i a t n i k o v, P. B., On a particular criterion of optimal time. PMM Vol. 40, No. 5, 1976.

10. H a r t m a n, P., Ordinary Differential Equations. New York, J. Wiley and Sons, Inc., 1964.

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